Minimizing the Cost of Precision
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This paper concerns the cost of adding precision to a product. Most commonly this occurs when precision is needed to make the product meet certain functional requirements. Increased precision typically adds to cost. Of interest is how to achieve the desired product functionality while minimizing cost. The method used is called variation analysis. To simplify the presentation and make it easier to follow, we present a simple example. Once you understand this example, you may find the method useful in more complex problems.

Consider a situation where an orifice will be used to closely control the amount of flow of a liquid in a machine. Hundreds of the machines will be built and sold. The rate of flow through the orifice Q is governed by the following equation which has been derived from accepted principles of fluid mechanics, and has been further validated by testing.

$$Q = 0.05d^2P^{1/2}$$

Here Q is the rate of flow in cubic feet per second (cfs), d is the diameter of the orifice in inches and P is the upstream pressure in pounds per square inch (psi). Note the assumption that only d and P control Q. If Q must be controlled accurately, then the accuracy of Q will in some sense depend on the accuracy of d and P.

Shortly we will explore how this can be evaluated. But first we take note of a well known phenomenon in industry—accuracy costs money. If a product is to be competitive we don't want any more accuracy than we need to satisfy customer goals.[1]

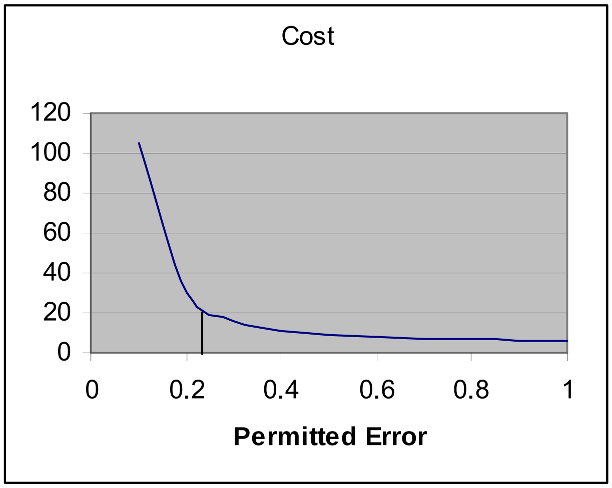
If the accuracy of more than one parameter must be considered in determining the accuracy of the product (as is the case with the orifice discharge problem we are considering), then a tradeoff may exist as to which parameters can most beneficially have looser tolerances in order to keep costs down. This will turn out to be the case with our orifice example.

[1] This notion goes against the grain with many design engineers, who are culturally prone to putting expensive accuracy requirements into drawings and specifications, apparently out of habit. Some even think this is good engineering.

The relationship between cost and accuracy can take several forms, anywhere from a simple straight line relationship to a curve with a knee as illustrated in Exhibit 1.

Note: the numbers on the axes in Exhibit 1 have no significance. They are for illustration only. When the cost vs. permitted error relationship has a pronounced knee as in Exhibit 1, it's important to try to keep the design to the right of the knee if at all possible. The important thing to note is that if we allow permitted error to be greater than about 0.23 (note the vertical line in the exhibit), costs remain below 20 dollars. And they change only gradually as tolerance of error changes. There's not much difference in cost between a permitted error of 0.5 and 0.6. But if permitted error is forced to go below 0.23, costs rise sharply. There is a huge difference in cost between a permitted error of 0.2 and 0.1. This is illustrative of the knee of the curve phenomenon that occurs in many tradeoffs of cost versus accuracy.[2]

Exhibit 1—Typical Curve with Knee



[2] The true knee (point of maximum curvature) is not quite where it appears to be in Exhibit 1. The axes in which the curve is plotted have different scales, causing the knee to appear to be offset somewhat from where it truly is. This matters little in this hypothetical discussion, but it could be relevant in an actual problem if the location of the true knee is important. The differential calculus provides a means for finding the point if maximum curvature. See almost any introductory calculus text.

Let us suppose that the nominal pressure we will work with is 10 pounds per square inch and that the nominal diameter of the orifice is 2 inches. The pressure will be controlled by a purchased pressure regulator and the orifice will be shaped by internal grinding of a hole in a steel plate. Based on the above formula the flow we will achieve with this diameter and this pressure will be:

$$Q = (0.05)(2^2)(10^{1/2}) = 0.63 \text{ cfs}$$

Let us further suppose that we need to keep the flow within ±0.0945 cfs (±15%) of this value in order to meet customer goals. This requirement clearly will drive the tolerances we must impose on d and on P, and will also drive the cost we must pay because accuracy is expensive. But we will want to keep the cost as low as possible.

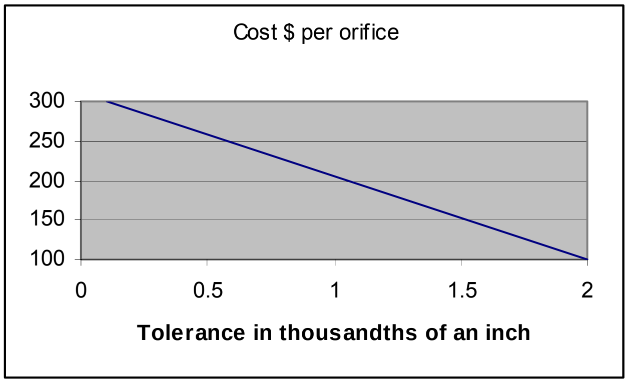
We find that tolerances as tight as ±0.002 inches for the orifice are readily obtained for $100 an orifice (this is hypothetical, of course).[3] We assume that looser tolerances do not significantly decrease cost. But for tolerances tighter than ±0.002 inches costs increase significantly. We find after a bit of research that the following linear equation closely approximates the cost effect for tighter tolerances (again, this is hypothetical).

Orifice cost $ = 310 – 105*(tolerance in thousandths of an inch)

Assume that the tightest practical tolerance is ±0.0001 inches, so our range of interest is from ±0.0001 inches to ±0.002 inches. Cost per orifice over this range of interest is plotted in Exhibit 2.

Exhibit 2—Cost of Orifice vs. Tolerance

[3] The American Society of Mechanical Engineers and other organizations publish research into machining costs and the cost effects of tighter tolerances.

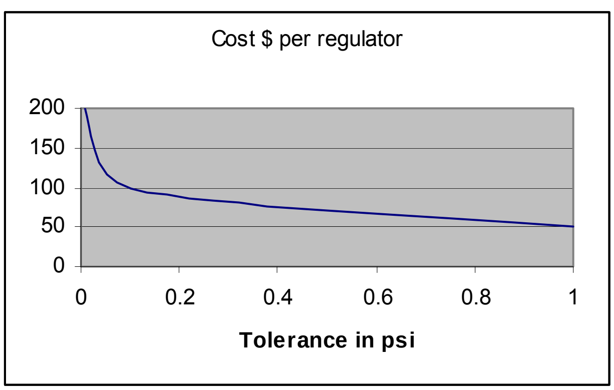


We next consider pressure regulators. Let us (hypothetically) suppose that a survey of vendors turns up suitable regulators with accuracy of regulation ranging from 0.1% to 10% of the desired pressure. After obtaining several vendor quotes and doing a regression analysis on the resulting data we determine that the following equation is a good fit to the data:

$$\text{Regulator cost \$} = 100*(10*\text{regulation error in psi})^{-0.3}$$

Exhibit 3 plots the regulator cost versus the error in psi.

Exhibit 3—Regulator Cost vs. Tolerance



Now we introduce the means for connecting the errors in the orifice and in the pressure regulator to the errors in the flow quantity. It is known to statisticians as the propagation of error formula. Although it probably is of most interest to engineers and scientists, it is not discussed in many statistics texts written specifically for engineers and scientists. Readers interested in further pursuing it may have to look in several books to find it. Our source is "Basic Statistical Methods for Engineers and Scientists" by Neville and Kennedy, second printing, published in 1966 by International Textbook Company, Scranton, PA.

Readers having some familiarity with statistical methods may recall that if two random variables are independent, then the variance of their sum is equal to the sum of their variances, thus:

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$$

This is a special case of the more general propagation of error formula that is not limited to simple sums. It can cope with many types of continuous functional relationships and can deal with two or more variables that contribute to error. The expression of this relationship we will use is:

$$\sigma_Q^2 = (\partial Q/\partial d)^2\, \sigma_d^2 + (\partial Q/\partial P)^2\, \sigma_P^2$$

The propagation of error formula as applied to our flow control example states that the variance in flow [σ_Q^2] is equal to the variance in orifice diameter [σ_d^2] multiplied by the square of the partial derivative of Q with respect to d [$(\partial Q/\partial d)^2$] plus the variance of P [σ_P^2] multiplied by the square of the partial derivative of Q with respect to P [$(\partial Q/\partial P)^2$].

As previously noted, we have a formula relating Q to d and P, namely:

$$Q = 0.05d^2P^{1/2}$$

Taking partial derivatives, and evaluating at the nominal diameter and pressure leads to:

$$\sigma_Q^2 = 0.4\, \sigma_d^2 + 0.001\, \sigma_P^2$$

Note the coefficients of the variance terms on the right hand side of this equation. The coefficient 0.4 is 4,000 times the size of the coefficient 0.001. This is a clue that perhaps the error in diameter of the orifice will be of much more significance than the error in regulated pressure. Mostly this is due to the fact that diameter is squared in the underlying equation, while the influence of pressure varies only as the square root. In general, the error effects of variables that are raised to higher powers or that are exponential in nature will have more impact than linear variables or variables that are raised to lower powers. This intuitively makes sense.

How do we evaluate the variances on the right hand side of the formula? We consider both d and P to be normally distributed with means 2 inches and 10 psi respectively, and consider their variances to be related to the tolerances we impose. For example, suppose we impose a tolerance of ±0.001 inches on the orifice diameter. Because of the assumption of a normal distribution, we can conveniently consider 0.001 inches to be equivalent to three standard deviations (see any decent introductory text on statistics). One standard deviation is therefore 0.001/3 = 0.00033 in. The variance is defined as the square of the standard deviation, therefore assigning a tolerance of ±0.001 inches results in a variance of $(0.00033)^2$ or 1.111E-7 in^2.

We can easily construct a simple formula for what we just demonstrated.

$$\sigma_d^2 = (T/3)^2$$

Here T is the one-sided tolerance we have imposed. By one-sided we mean that if we have imposed ±0.001 inches, then T = 0.001. This formula is reasonable for the assumption of a normally distributed error. It is almost as accurate for many non-normal distributions that occur in practice.

We can now rewrite our previous equation in terms of assigned tolerances:

$$\sigma_Q^2 = 0.4\ (T_d/3)^2 + 0.001\ (T_P/3)^2$$

We are ultimately interested not in the variance of Q, but in its standard deviation. We can obtain that by taking the square root of the above. At the same time, we convert σ_Q to its tolerance equivalent using $\sigma_Q = T_Q/3$:

$$T_Q = 3[0.4\ (T_d/3)^2 + 0.001\ (T_P/3)^2]^{1/2}$$

Let’s define two additional variables:

$$C_d = \text{cost of one orifice of diameter } d$$

C_P = cost of one pressure regulator

We can now write two previously developed equations in terms of our newly defined variables, and create yet another useful equation, this time for the total cost, C_{tot}:

$$C_d = 310 – 105T_d$$

$$C_P = 100(10T_P)^{-0.3}$$

$$C_{tot} = C_d + C_P = 310 – 105T_d + 100(10T_P)^{-0.3}$$

Let's summarize what we have done to this point. We have developed the following two equations that we will use to minimize total cost:

$$T_Q = 3[0.4\ (T_d/3)^2 + 0.001\ (T_P/3)^2]^{1/2}$$

$$C_{tot} = 310 – 105T_d + 100(10T_P)^{-0.3}$$

We have also established the following constraint:

$$0 \leq T_Q \leq 0.0945 \text{ cfs}$$

Additionally we have established the following practical ranges that are effectively constraints:

$$0.1 \leq T_d \leq 2 \text{ thousandths of an inch}$$

$$0.01 \leq T_P \leq 1 \text{ psi}$$

What we now have is a rather complicated constrained non-linear optimization problem in which we want to minimize C_{tot} subject to constraints on T_Q, T_d, and T_P. Elegant techniques exist for solving such problems, and in a more complex problem than this one their use may be warranted. Here we will first use a very inelegant approach. We will create a spreadsheet containing both the T_Q equation and the C_{tot} equation and proceed by trial and error by entering various values of T_d and T_P within their allowable ranges. There is no guarantee that this method will find an absolute minimum but as it happened it can come very close with a few minutes of exploration of the trade space.

In the particular problem we are dealing with, the cost is much more sensitive to orifice diameter than to pressure control. Our intuitive trial and error search strategy begins by using the cheapest regulator then tightening the diameter tolerance until the constraint on Q is met. Then we explore that vicinity until the constraint is satisfied and we get the lowest cost we can find. Exhibit 4 shows the simple spreadsheet we used in this search, and the solution we quickly found.

Our second approach was to use Monte Carlo simulation in a spreadsheet to rapidly try many combinations of tolerances. This approach is essentially computer-aided trial and error. The best solution we found in 5,000 simulations was less than a dollar away from our intuitive solution, but when a problem gets much more complicated than this simple example Monte Carlo will be far more reliable and faster than an intuitive search.

The Monte Carlo simulations ran in less than a minute. We did have to spend about an hour programming a macro in the VBA language to automate the Monte Carlo.

Exhibit 4—Manual Search Spreadsheet

Minimization of total cost

0.14	$=T_d$ (0.1-2 thousandths in)
1	$=T_P$ (0.01-1 psi)
0.094021	$=T_Q$ ($\le$0.0945 cfm)
$345	$=C_{tot}$ $